# 行列式计算作业—一题多解

18-3）

$$D_n = \begin{vmatrix} \alpha+\beta & \alpha\beta & 0 & \cdots & 0 & 0 \\ 1 & \alpha+\beta & \alpha\beta & \cdots & 0 & 0 \\ 0 & 1 & \alpha+\beta & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & \alpha+\beta \end{vmatrix}_{n\times n} = \frac{\alpha^{n+1}-\beta^{n+1}}{\alpha-\beta},\ (\alpha\neq\beta)$$

$$(D_n = (n+1)\alpha^n, \alpha = \beta)$$

证法 1：第二数学归纳法

$$D_1 = \alpha+\beta = \frac{\alpha^2-\beta^2}{\alpha-\beta}$$

$$D_2 = \begin{vmatrix} \alpha+\beta & \alpha\beta \\ 1 & \alpha+\beta \end{vmatrix} = \frac{\alpha^3-\beta^3}{\alpha-\beta}$$

等式成立。假设等式在 $n \leq k$ 的情况都成立。将 $D_n$ 按第一列展开，有

$$D_n = (\alpha+\beta)D_{n-1} - \alpha\beta D_{n-2} = (\alpha+\beta)\frac{\alpha^n-\beta^n}{\alpha-\beta} - \alpha\beta\frac{\alpha^{n-1}-\beta^{n-1}}{\alpha-\beta} = \frac{\alpha^{n+1}-\beta^{n+1}}{\alpha-\beta}$$

命题得证！ □

证法 2：第一数学归纳法（From–吴文众同学）

$n = 1$ 时等式成立，

$$D_1 = \alpha+\beta = \frac{\alpha^2-\beta^2}{\alpha-\beta}$$

假设 $n-1$ 时等式成立，

$$D_{n-1} = \frac{\alpha^n-\beta^n}{\alpha-\beta} = \alpha^{n-1} + \alpha^{n-2}\beta + \cdots + \alpha\beta^{n-2} + \beta^{n-1}$$

则对于 $D_n$ 有

$$D_n = \begin{vmatrix} \alpha & \alpha\beta & 0 & \cdots & 0 & 0 \\ 1 & \alpha+\beta & \alpha\beta & \cdots & 0 & 0 \\ 0 & 1 & \alpha+\beta & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & \alpha+\beta \end{vmatrix}_A + \begin{vmatrix} \beta & \alpha\beta & 0 & \cdots & 0 & 0 \\ 0 & \alpha+\beta & \alpha\beta & \cdots & 0 & 0 \\ 0 & 1 & \alpha+\beta & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & \alpha+\beta \end{vmatrix}_B$$

对第一个行列式 A，从第 2 列开始每列减去前一列的 $\beta$ 倍，有

$$A = \begin{vmatrix} \alpha & 0 & 0 & \cdots & 0 & 0 \\ 1 & \alpha & 0 & \cdots & 0 & 0 \\ 0 & 1 & \alpha & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & \alpha \end{vmatrix} = \alpha^n$$

第二个行列式 $B = \beta D_{n-1}$。所以

$$D_n = \alpha^n + \beta D_{n-1} = \alpha^n + \beta \times (\alpha^{n-1} + \alpha^{n-2}\beta + \cdots + \alpha\beta^{n-2} + \beta^{n-1}) = \frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta}$$

命题得证！ □

证法 3：直接求解（特征方程–From 张海齐, 付邦营同学）

将 $D_n$ 按第一列展开

$$D_n = (\alpha+\beta)D_{n-1} - \alpha\beta D_{n-2}$$

构造等比数列

$$D_n - \alpha D_{n-1} = \beta^n$$

$$D_n - \beta D_{n-1} = \alpha^n$$

可解出 $D_n$.

（或由

$$D_n = (\alpha+\beta)D_{n-1} - \alpha\beta D_{n-2}$$

有特征方程 $\lambda^2 - (\alpha+\beta)\lambda + \alpha\beta = 0$. 方程的根为 $\lambda_1 = \alpha, \lambda_2 = \beta$. 则可设

$$D_n = c_1\alpha^n + c_2\beta^n$$

代入 $D_1, D_2$ 解得 $c_1 = \frac{\alpha^n}{\alpha-\beta}, c_2 = \frac{\beta}{\beta-\alpha}$. 故

$$D_n = \frac{\alpha}{\alpha-\beta} \times \alpha^n + \frac{\beta}{\beta-\alpha} \times \beta^n = \frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta}$$

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补充题 3-1）

$$\begin{vmatrix} a_{11}+x & a_{12}+x & \cdots & a_{1n}+x \\ a_{21}+x & a_{22}+x & \cdots & a_{2n}+x \\ \vdots & \vdots & & \vdots \\ a_{n1}+x & a_{n2}+x & \cdots & a_{nn}+x \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} + x\sum_i\sum_j A_{ij}$$

证法一：拆分法

$$\begin{vmatrix} a_{11}+x & a_{12}+x & \cdots & a_{1n}+x \\ a_{21}+x & a_{22}+x & \cdots & a_{2n}+x \\ \vdots & \vdots & & \vdots \\ a_{n1}+x & a_{n2}+x & \cdots & a_{nn}+x \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12}+x & \cdots & a_{1n}+x \\ a_{21} & a_{22}+x & \cdots & a_{2n}+x \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2}+x & \cdots & a_{nn}+x \end{vmatrix}_A + \begin{vmatrix} x & a_{12}+x & \cdots & a_{1n}+x \\ x & a_{22}+x & \cdots & a_{2n}+x \\ \vdots & \vdots & & \vdots \\ x & a_{n2}+x & \cdots & a_{nn}+x \end{vmatrix}_B$$

后一行列式 B 第 2、3、$\cdots$、n 列减去第 1 列，然后按第 1 列展开有：

$$\begin{vmatrix} x & a_{12}+x & \cdots & a_{1n}+x \\ x & a_{22}+x & \cdots & a_{2n}+x \\ \vdots & \vdots & & \vdots \\ x & a_{n2}+x & \cdots & a_{nn}+x \end{vmatrix} = \begin{vmatrix} x & a_{12} & \cdots & a_{1n} \\ x & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ x & a_{n2} & \cdots & a_{nn} \end{vmatrix} = x\sum_i A_{i1}$$

类似不断进行下去，行列式 $A$ 依次拆分第 2、3、$\cdots$、n 列，命题得证。 □

证法二：升阶法 (From 吴振铭同学)

$$\begin{vmatrix} a_{11}+x & a_{12}+x & \cdots & a_{1n}+x \\ a_{21}+x & a_{22}+x & \cdots & a_{2n}+x \\ \vdots & \vdots & & \vdots \\ a_{n1}+x & a_{n2}+x & \cdots & a_{nn}+x \end{vmatrix}_{n\times n} = \begin{vmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & a_{11}+x & a_{12}+x & \cdots & a_{1n}+x \\ 1 & a_{21}+x & a_{22}+x & \cdots & a_{2n}+x \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & a_{n1}+x & a_{n2}+x & \cdots & a_{nn}+x \end{vmatrix}_{(n+1)\times(n+1)}$$

从第 2 列开始，每列减去第 1 列的 $x$ 倍，得

$$\begin{vmatrix} 1 & -x & -x & \cdots & -x \\ 1 & a_{11} & a_{12} & \cdots & a_{1n} \\ 1 & a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}_{(n+1)\times(n+1)}$$

按第一行展开可证。 □