An integral homology of Coxeter cellular complexes

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Content

1. Orbifold, Coxeter orbifold
2. Orbifold homology of Coxeter cellular complexes
3. Examples

What is an orbifold?

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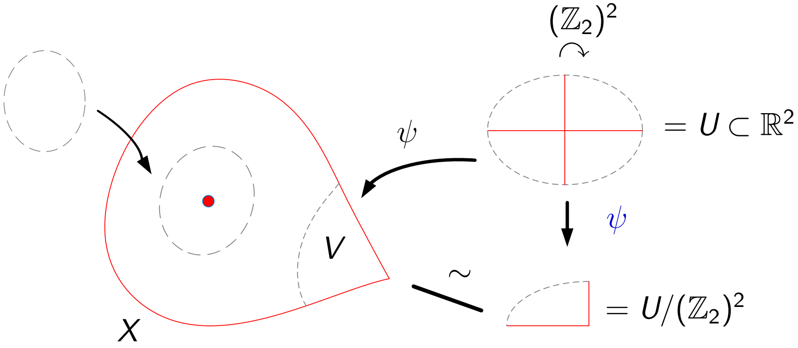

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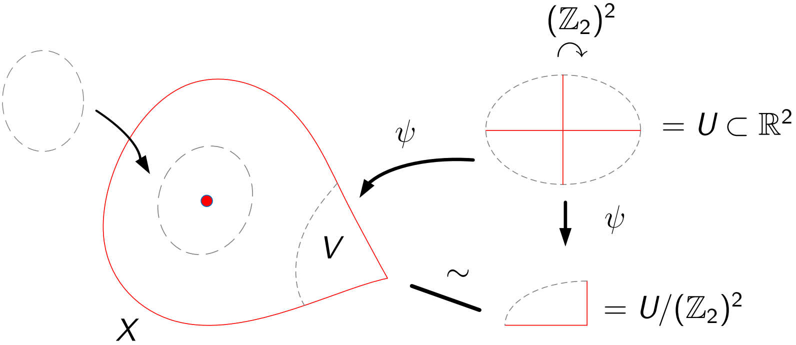
- Local group, underlying space ($|X|$).



What is a Coxeter orbifold?

♣ *Coxeter group*

$$W = \langle s_1, s_2, \cdots, s_m \mid (s_i s_j)^{m_{ij}} = 1, \forall 1 \leq i \leq j \leq m \rangle$$

where $m_{ii} = 1$ and $m_{ij} \geq 2$ for $i \neq j$.

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♣ Coxeter orbifolds $\Rightarrow$ manifolds with corners (Davis).

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How to define orbifold homolgy groups?

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$$H^*_{CR}(X;\mathbb{R}) := \bigoplus_{(g)\in T} H^{i-2I(g)}(X_{(g)};\mathbb{R})$$

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- (BNSS) Integral homology of q-CW complexes with cells in even dimensions.

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e^n is called the blow-up of e^n/W.
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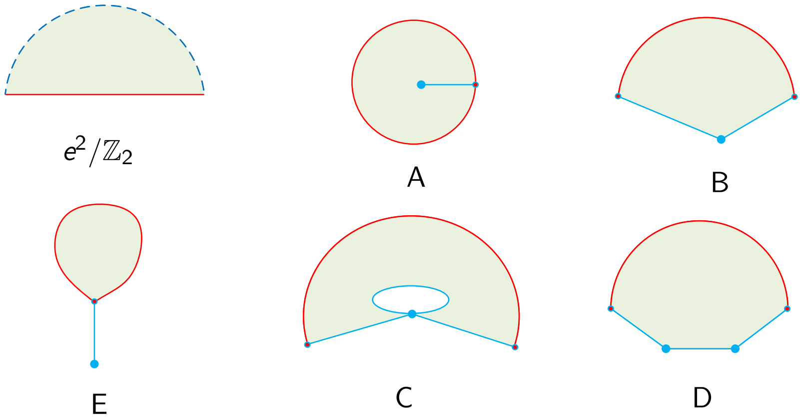
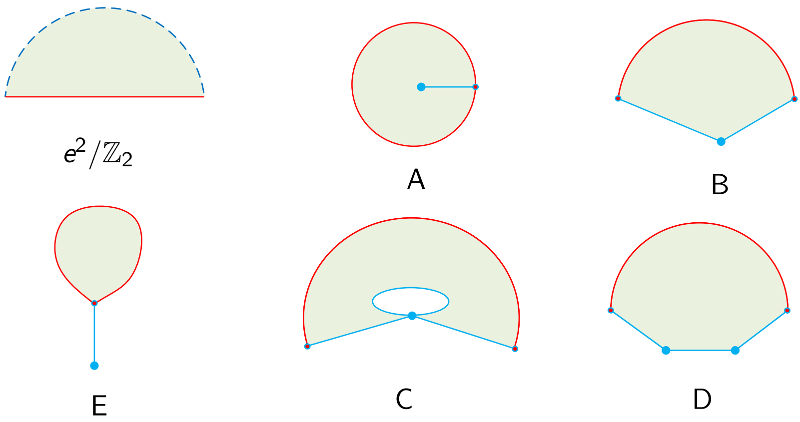
♣ The boundary of a regular cell contains no singular cell.

An example





($\star$) E is a q-CW complex but not a Coxeter cellular complex.

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Orbifold cellular homology of Coxeter cellular complexes

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How to define boundary maps?

$d : C_n \longrightarrow C_{n-1}$ is defined by their blow-up, which should be determined by ϕ, ψ, ∂.

$$\begin{array}{ccccc} \overline{e^n} & \xrightarrow{\partial} & \partial\overline{e^n} = S^{n-1} & \xrightarrow{\psi} & \partial\overline{e^n}/W \\ \Big\downarrow{\scriptstyle \psi} & & & & \Big\downarrow{\scriptstyle \phi} \\ \overline{e^n}/W & & \xrightarrow{\quad\quad\Phi\quad\quad} & & X^{n-1} \end{array}$$

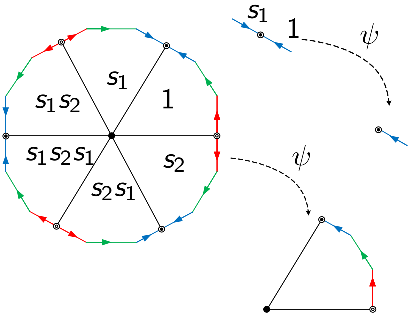
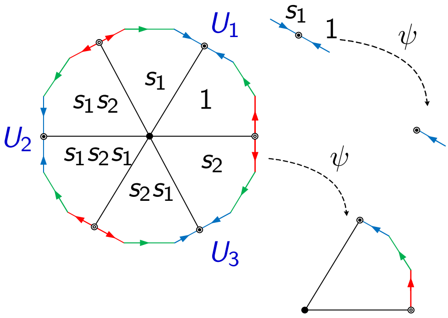


Figure: $W = < s_1, s_2 \mid s_1^2 = s_2^2 = (s_1 s_2)^3 = 1 >$



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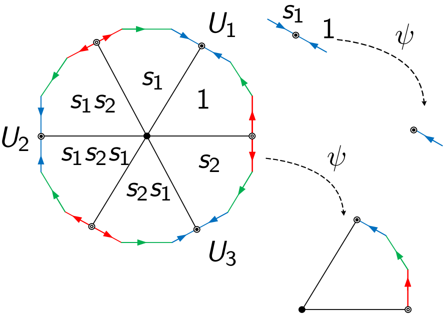

Cosets		W_β
$U_1 = \{1, s_1\}$	$\searrow$	
$U_2 = \{s_1 s_2, s_1 s_2 s_1\}$	$\longrightarrow$	$\{1, s_1\}$
$U_3 = \{s_2, s_2 s_1\}$	$\nearrow$	

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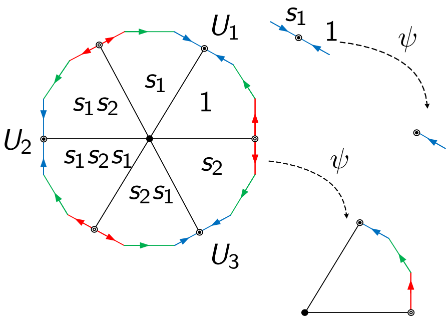

Cosets $\qquad$ W_β

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Key point: The presentation of coset U_i determines $U_i \to W_\beta$.

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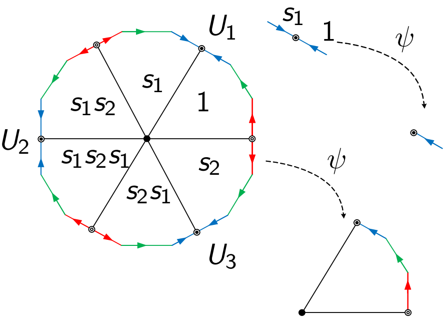

Cosets — W_β

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E.g. $U_2 = s_1 s_2 \cdot \{1, s_1\} = s_1 s_2 s_1 \cdot \{s_1, 1\}$.

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Remark

Another rule of g: Chosen with even word length if possible.
The associated orbifold homology is related to weight homology under some special cases.

ϕ is still trivial, $d_\beta = ?$

$$
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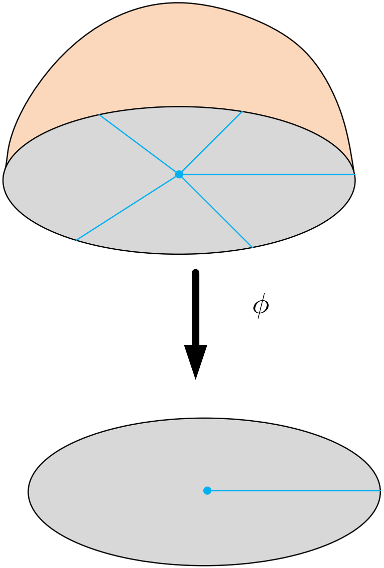
If $\#(S(W) - S(W_\beta)) \geq 2$, then $\frac{|W|}{|W_\beta|}$ always even. So $d_\beta = 0$.

When ψ is trivial. $(W = W_\beta)$



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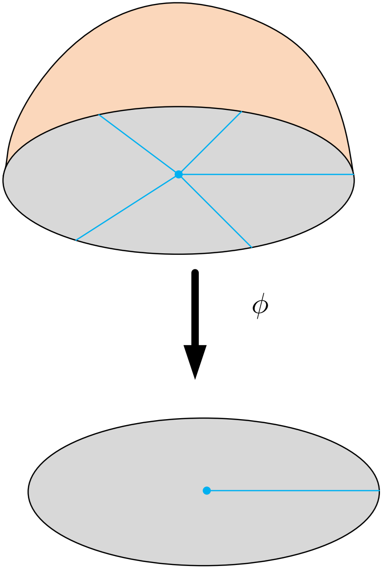

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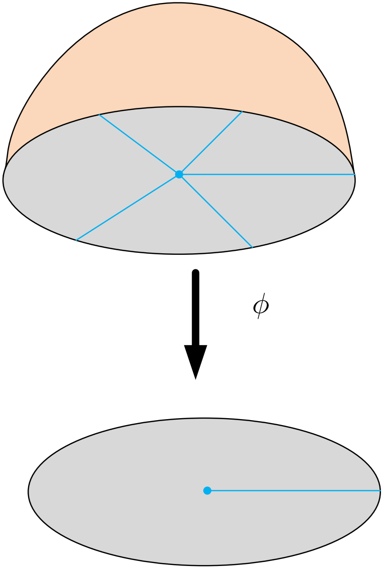

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the degree of $S^1 \longrightarrow S^1$.

Boundary map

$$d(e^n/W) = \sum_{\#S(W_\beta) \geq \#S(W)-1} n_\beta \left(\frac{|W|}{|W_\beta|} \mod 2 \right) e_\beta^{n-1}/W_\beta.$$

Boundary map

The boundary map $d : C_n \longrightarrow C_{n-1}$ is defined by their blow-ups,

$$d(e^n/W) \triangleq [\phi \circ \psi(\sum_{g \in W} (-1)^{l(g)} \partial(\overline{e^n} \cap \mathcal{X}_g))]$$

where

- *$l(g)$ is the word length of g in W;*
- *$\mathcal{X}_g$ is a lifting of $\phi^{-1}(e^{n-1}_\beta / W_\beta)$ in $\overline{e^n}/W$, that is, the coset indexed by g.*

Remark

It is valid in some non-Coxeter orbifolds.

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$$T := \{\text{All faces of } X\}/\sim$$

Theorem

$$H_i^{orb}(X) = \bigoplus_{J \in T} H_{i-l(J)}(X_J) \tag{1}$$

where $l(J)$ is the codimension of the highest dimensional face in J.

Remark

Which is an analogue of Chen-Ruan cohomology groups and Hochster's formula.

Something different (Under $\mathbb{Z}$-coefficient)

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$$H_n^{orb}(S^n/W) \ncong \mathbb{Z}.$$

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Hurewize theorem

$$\left(\pi_1^{orb}(X)\right)^{ab} \cong H_1(|X|) \oplus H_1(X_{sing}, \mathbb{Z}_2).$$

Homology properties

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The long exact sequence of pair, homotopic invariant, universal coefficient theorem.

Example (Simple polytope)

Let P be a simple polytope equipped with a right-angled Coxeter orbifold structure. Then the standard cubical decomposition of P is a (righ-angled) Coxeter cellular decomposition of P. Then

$$\pi_1^{orb}(P) \cong \pi_1^{orb}(P^2) = W_P$$

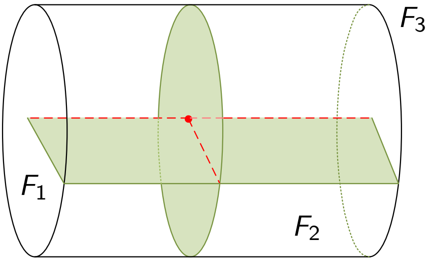
where W_P is the right-angled Coxeter group of P.

$$H_i^{orb}(P) = \mathbb{Z}^{f_{n-i}} \tag{2}$$

where f_{n-i} is the number of $(n-i)$-faces of P.

Example (Coxeter cylinder)

Let Q be a solid cylinder with three facets F_1, F_2 and F_3. $F_1 \cap F_2$ is labelled by 2, and $F_1 \cap F_2$ is labelled by 3. Then Q is a Coxeter orbifold. Q can be decomposed to one 0-cell, three 1-cells, three 2-cells and two 3-cells.



$$X_1 \cong D^3, X_{[s_1]} = F_1 \cong D^2$$

$$X_{[s_2]} = F_2 \cup F_3 \cup (F_2 \cap F_3) \cong D^2$$

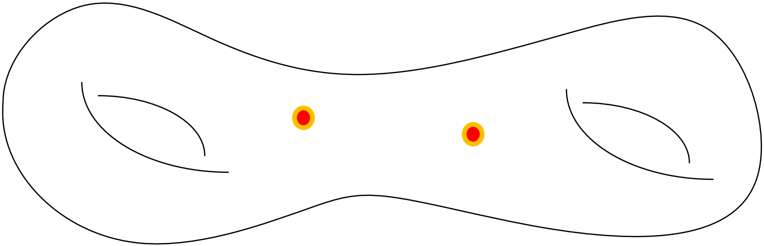
$$X_{[s_1 s_2]} = F_1 \cap F_2 \cong S^1$$

$$H_i^{orb}(Q) = \begin{cases} \mathbb{Z}, & i = 0, 2, 3 \\ \mathbb{Z}^2, & i = 1 \\ 0, & otherwise. \end{cases}$$

Figure: Coxeter cylinder

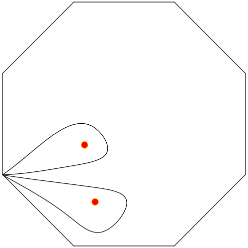
Example (Surface with isolated singular points)

Let S is a closed genus 2 orientable surface with two isolated singular points $\{v_1, v_2\}$, each v_i has a local group $\mathbb{Z}_{n_i}$ generated by a rotation.



Example-3

Then we can give a cell decomposition of S.



$$\pi_1^{orb}(S) = \langle x_1, y_1, x_2, y_2, s_1, s_2 \mid s_1^{n_1} = s_2^{n_2} = 1, [x_1, y_1][x_2, y_2]s_1 s_2 = 1 \rangle$$

the homology groups of S,

$$H_i^{orb}(S) \cong \begin{cases} \mathbb{Z}, & i = 0, 2 \\ \mathbb{Z}^4 \oplus \mathbb{Z}/(n_1, n_2)\mathbb{Z}, & i = 1 \\ 0, & \textit{otherwise}. \end{cases}$$

Thank You

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Some references

- Lü, Yu and Wu, *An integral homology of Coxeter orbifolds*, preprint, 2020.
- Adem, Leida and Ruan, *Orbifolds and stringy topology*, 2007.
- Bahri, Notbohm, Sarkar and Song, *On integral cohomology of certain orbifolds*. IMRN.
- Poddar and Sarkar,*On quasitoric orbifolds*, 2010.
- Weimin Chen and yongbin Ruan, *A new cohomology theory of orbifold*, 2004.
- Satake, *On a generalization of the notion of manifold*, 1956.
- Satake, *The Gauss-Bonnet Theorem for V-manifolds*, 1957.
- Allen Hatcher, *Algebraic topology*, 2002.

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