Integral homology groups of Coxeter orbifolds

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Content

1. Orbifold, Coxeter orbifold
2. Orbifold homology of Coxeter cellular complexes

What is an orbifold?

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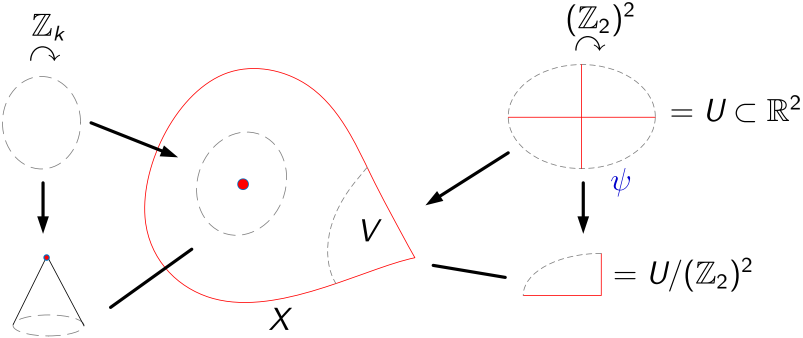
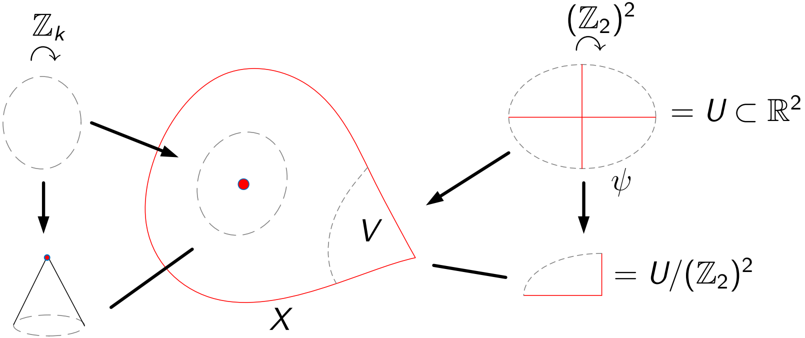


What is an orbifold?

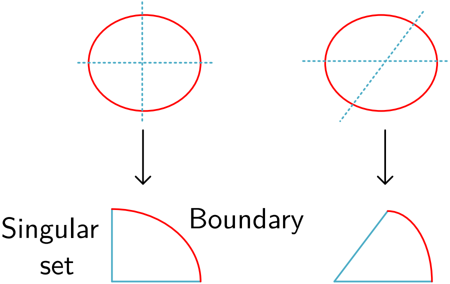
- An *n*-orbifold is a singular space locally modelled on quotients of an open set of $\mathbb{R}^n$ by a finite group action.
- Local group, underlying space ($|X|$).



Example

Example

Let M be a manifold, G a discrete group acting properly on M, then the isotropy subgroup G_x is finite for any $x \in M$. Now the orbit space M/G is an orbifold, which is called a **quotient orbifold**.



$$G_1 =< s_1, s_2 \mid s_1^2 = s_2^2 = (s_1 s_2)^2 = 1 >$$
$$G_2 =< s_1, s_2 \mid s_1^2 = s_2^2 = (s_1 s_2)^3 = 1 >$$

What is a Coxeter orbifold?

♣ *Coxeter group*

$$W = \langle s_1, s_2, \cdots, s_m \mid (s_i s_j)^{m_{ij}} = 1, \forall 1 \leq i \leq j \leq m \rangle$$

where $m_{ii} = 1$ and $m_{ij} \geq 2$ for $i \neq j$.

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♣ (Coxeter 1935) Every finite Coxeter group has a representation as a reflection group of $\mathbb{R}^n$, and $\mathbb{R}^n / W \cong Cone^m \times \mathbb{R}^{n-m}$.

♣ A *Coxeter n-orbifold* is an orbifold locally modelled on $\mathbb{R}^n / W$ where W is a finite Coxeter group.

How to define orbifold homolgy groups?

- de Rham cohomology, [Satake, 1956].

$$H^*_{dR}(X;\mathbb{R}) \cong H^*(|X|;\mathbb{R})$$

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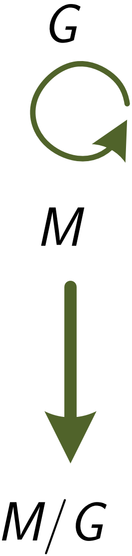
- Chen-Ruan cohomology of almost complex orbifolds, [CR, 2004],

$$H^*_{CR}(X;\mathbb{R}) := \bigoplus_{(g)\in T} H^{i-2l(g)}(X_{(g)};\mathbb{R})$$

  where $X_{(1)} = X$ non-twisted sector, $X_{(g)}$ for $g \neq 1$ twisted sector.

Borel construction and equivariant homology



G

M

M/G

$$\begin{tikzcd}
G \arrow[dr, bend left] & & G \arrow[dr, bend left] & \\
& M \arrow[rr, "\simeq"] \arrow[d, "\pi"'] & & EG\times M \arrow[d, "\pi"'] \\
& M/G & & (EG\times M)/G
\end{tikzcd}$$

Borel space: $EG\times_G M \triangleq (EG\times M)/G$

Equivariant (co)homology of G-space M,

$$H_*^G(M) \triangleq H_*(EG\times_G M) \quad \Rightarrow H_*^{orb}(M/G)$$

$$H_G^*(M;R) \triangleq H^*(EG\times_G M;R) \quad \Rightarrow H^*_{orb}(M/G;R)$$

How to define integral orbifold homolgy groups?

- Orbifold singular homology, [Takeuchi-Yokoyama, 2006-2012].

$$s\text{-}H_*(X) \cong H_*(|X|)$$

$$t\text{-}H_*(X;\mathbb{Q}) \cong H_*(|X|;\mathbb{Q})$$

  where *t*-singular homology with $\mathbb{Z}$-coefficient can capture the orbifold structure.

- Orbifold cellular homology, [PS, 2010] & [BNSS, 2019].

q-CW complex (Poddar-Sarkar, Bahri-Nothbohm-Sarkar-Song)

- q-cell: e^n/G, where $|G| < \infty$.
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- (PS) Def of H_*^{orb}: using the property

$$H_p\Big((X_n, X_{n-1}); \mathbb{Q}\Big) = \bigoplus \widetilde{H}_p\Big(\frac{D^n/G}{\partial(D^n/G)}; \mathbb{Q}\Big)$$

$$\widetilde{H}_p\Big(\frac{D^n/G}{\partial(D^n/G)}; \mathbb{Q}\Big) = \begin{cases} H_{p-1}(S_\alpha^{n-1}/G_\alpha; \mathbb{Q}), & p \geq 2 \\ 0, & \text{otherwise} \end{cases}$$

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- (BNSS) Integral homology of q-CW complexes with cells in even dimensions.
  Witout explicit boundary map.

Special q-CW complex: Coxeter cellular complex

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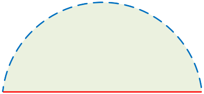
- ♣ Coxeter cell: e^n/W where W is a finite Coxeter group.

- ♣ Coxeter cellular complex: Every attaching map

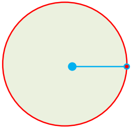
$$\phi : \partial\overline{e^n}/W \to X^{n-1}$$

preserves the local groups.

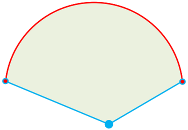
Example



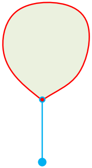
$e^2/\mathbb{Z}_2$



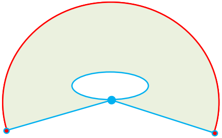
A



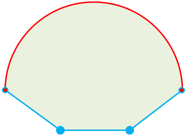
B



E

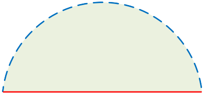


C



D

Example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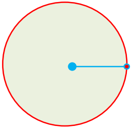
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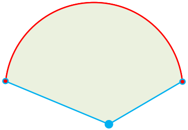
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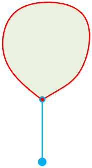
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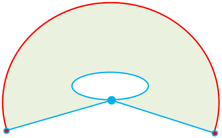
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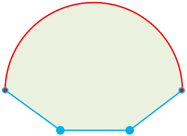
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($\star$) E is a q-CW complex but not a Coxeter cellular complex.

Orbifold cellular homology of Coxeter cellular complexes

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$$\begin{tikzcd}
\overline{e^n} \arrow[r, "\partial"] \arrow[d, "\psi"'] & \partial\overline{e^n} = S^{n-1} \arrow[r, "\psi"] & \partial\overline{e^n}/W \arrow[d, "\phi"] \\
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Definition (Boundary map of Coxeter cellular complex)

$$d(e^n/W) = \sum n_\beta \Theta\left(\frac{|W|}{|W_\beta|}\right) e_\beta^{n-1}/W_\beta$$

where $\Theta\left(n\right) = \begin{cases} 1, & n \text{ odd} \\ 0, & n \text{ even} \end{cases}$.

When ϕ is trivial.

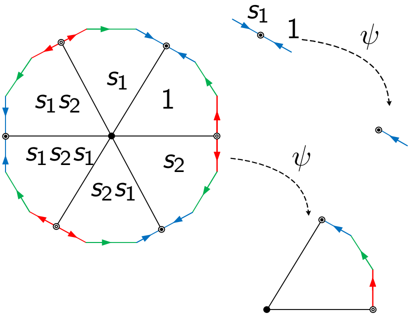

Figure: $W =< s_1, s_2 \mid s_1^2 = s_2^2 = (s_1 s_2)^3 = 1 >$

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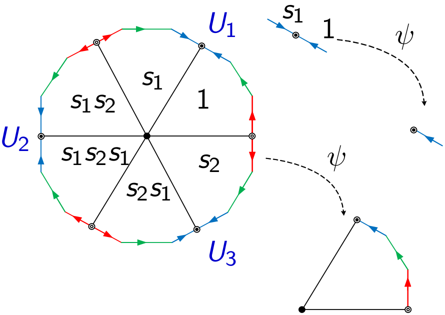

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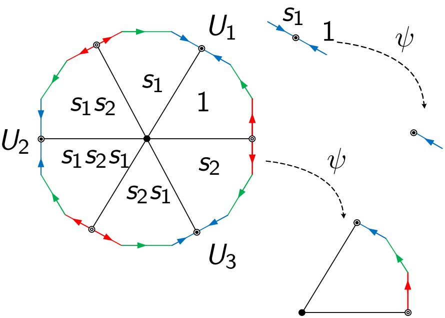

Cosets $\qquad W_\beta$

$$\begin{tikzcd}
U_1 = \{1, s_1\} \arrow[dr] & \\
U_2 = \{s_1 s_2, s_1 s_2 s_1\} \arrow[r] & \{1, s_1\} \\
U_3 = \{s_2, s_2 s_1\} \arrow[ur] &
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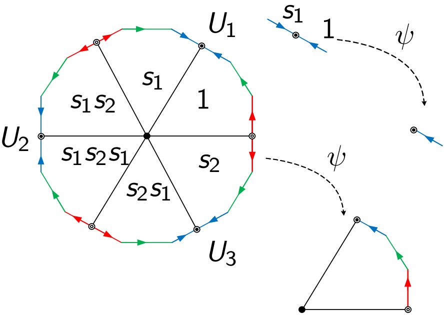

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Figure: $W =< s_1, s_2 \mid s_1^2 = s_2^2 = (s_1 s_2)^3 = 1 >$

Key point: The presentation of coset U_i determines $U_i \to W_\beta$.

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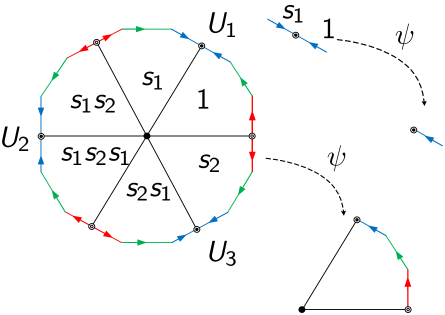

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E.g. $U_2 = s_1 s_2 \cdot \{1, s_1\} = s_1 s_2 s_1 \cdot \{s_1, 1\}$.

The rule of presentation g of each coset

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Each g is chosen with the **shortest word length**.

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Then each coset contributes a $(-1)^{\mathbf{l(g)}}[e^{n-1}/W_\beta]$ to $d_\beta\left[e_\beta^{n-1}/W_\beta\right]$, where $l(g)$ is the word length of reduced g in W.

ϕ is still trivial, $d_\beta = ?$

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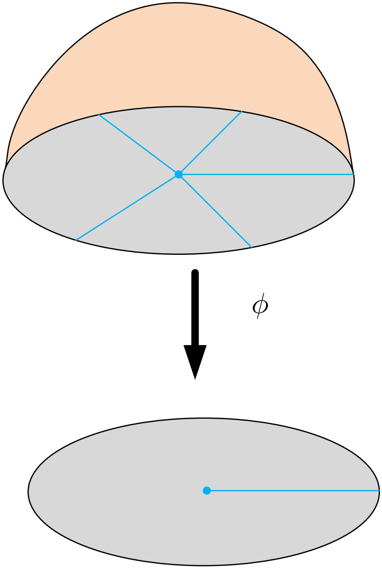
$\Rightarrow \Theta$.

When ψ is trivial ($W = W_\beta$).



$$\phi : \partial D^3/\mathbb{Z}_2 \longrightarrow X^2$$

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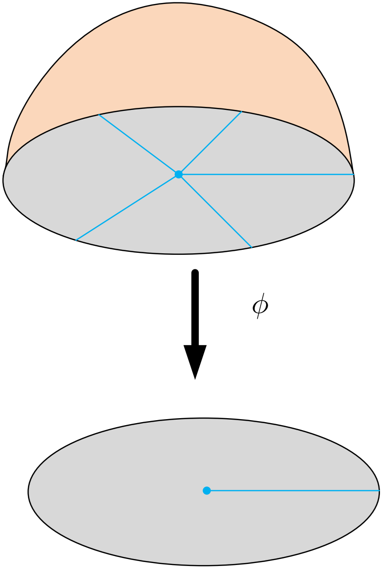

$\phi : \partial D^3/\mathbb{Z}_2 \longrightarrow X^2$

$d_\beta = n_\beta$

the degree of $S^1 \longrightarrow S^1$.

Boundary map

$d : C_n \to C_{n-1},$

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$d^2 = 0 \ \Rightarrow \ H_*^{orb}$

H_* of Coxeter cellular complexes

Theorem (Lü-Wu-Yu, 2021)

Let X be a Coxeter cellular complex, then

$$H_i^{orb}(X) = \bigoplus_{J \in T} H_{i-l(J)}(X_J)$$

where $l(J)$ is the codimension of the highest dimensional face in J.

Remark

Which is an analogue of Chen-Ruan cohomology groups and Hochster's formula.

Something different (Under $\mathbb{Z}$-coefficient)

$$H_i^{orb}(D^n/W, \partial(D^n/W)) \ncong H_i^{orb}\left(\frac{D^n/W}{\partial(D^n/W)}\right)$$

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In some cases,

$$H_n^{orb}(D^n/W, \partial(D^n/W)) \ncong H_{n-1}^{orb}(\partial(D^n/W))$$

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$$H_i^{orb}(D^n/W, \partial(D^n/W)) \ncong H_i^{orb}\left(\frac{D^n/W}{\partial(D^n/W)}\right)$$

In some cases,

$$H_n^{orb}(D^n/W, \partial(D^n/W)) \ncong H_{n-1}^{orb}(\partial(D^n/W))$$

In some cases,

$$H_n^{orb}(S^n/W) \ncong \mathbb{Z}$$

Homology properties

$$H_i^{orb}(D^n/W, \partial(D^n/W)) \cong \begin{cases} \mathbb{Z}, & i = n \\ 0, & otherwise \end{cases}$$

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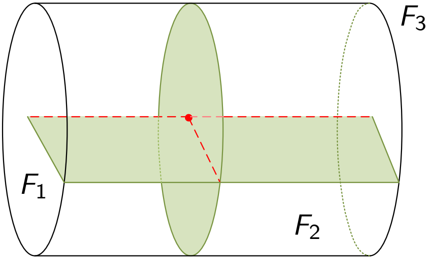
New simplicial/singular homology ($\neq t, s$-H_* by TY); characteristic class; etc.

Example

Example (Coxeter cylinder)

Let Q be a solid cylinder with three faces F_1, F_2 and F_3. $F_1 \cap F_2$ is labelled by 2, and $F_2 \cap F_3$ is labelled by 3. Then Q is a Coxeter orbifold. Q can be decomposed to one 0-cell, three 1-cells, three 2-cells and two 3-cells.



$$X_1 \cong D^3, X_{[s_1]} = F_1 \cong D^2$$

$$X_{[s_2]} = F_2 \cup F_3 \cup (F_2 \cap F_3) \cong D^2$$

$$X_{[s_1 s_2]} = F_1 \cap F_2 \cong S^1$$

$$H_i^{orb}(Q) = \begin{cases} \mathbb{Z}, & i = 0, 2, 3 \\ \mathbb{Z}^2, & i = 1 \\ 0, & otherwise \end{cases}$$

Figure: Coxeter cylinder

Thank You

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