The homology groups of colored polytopes

Li-Su Wu (*joint with* Fangbai Yang)

Shandong University of Science and Technology

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Small cover

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such that

$$\forall\ f = F_1 \cap F_2 \cap \cdots \cap F_i,$$
$$G_f \triangleq L(\lambda(F_1), \lambda(F_2), \cdots, \lambda(F_i)) \cong (\mathbb{Z}_2)^i. \qquad (\star)$$

Colored polytope

- Simple n-polytope P and characteristic map λ determine a small cover

$$M = P \times \mathbb{Z}_2^n / \sim$$

where $(p, g) \sim (q, h)$ iff $p = q$, $g^{-1}h \in G_{f(p)}$, and $f(p)$ is the unique face of P that contains p in its relative interior, $G_{f(p)} = \{1\}$ if $p \in P^\circ$.

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- In general, simple n-polytope P and characteristic map $\lambda : \mathcal{F} \longrightarrow \mathbb{Z}_2^k$ satisfying condition $(\star)$ determine a closed n-manifold in the same way.
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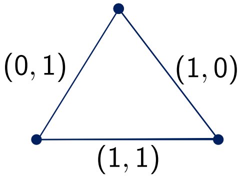
- Colored polytope P_λ:=simple n-polytope P + characteristic map λ. (Condition $(\star)$ is not essential.)
- The characteristic map λ is called a *colored map*, $\lambda(F)$ is called a *color* on facet F.

Homology of colored polytope

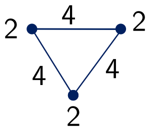
- Weighted homology.
  P_λ : colored polytope; K: dual complex of P_λ.
  Each simplex $\sigma_f \in K$ endowing with weight $w(\sigma_f) = |G_f|$.

$$\sigma_f \subset \sigma_{f'} \Leftrightarrow f' \subset f \Rightarrow G_f \subset G_{f'} \Rightarrow |G_f| \Big| |G_{f'}|$$

Hence K is a weighted complex.



colored polytope



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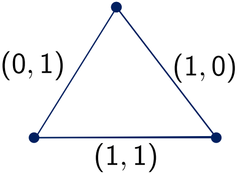
Hence K is a weighted complex. Weighted boundary operator

$$d_n^{wt}(\sigma) = \sum_{i=0}^{n} \frac{w(\sigma)}{w(\hat{\sigma}_i)} \cdot (-1)^i \hat{\sigma}_i.$$



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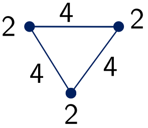
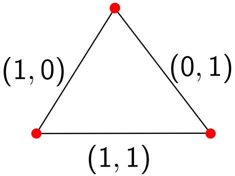


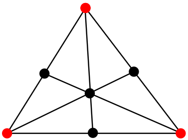
weighted complex

- Stratified simplicial homology (st-homology).
  P_λ : colored polytope; K: a triangulation adapted to P_λ.

$$H_i^{st}(K) = H_i^{wt}(K, \Sigma K)$$



Colored polytope



Triangulation

Faces of colored polytope

- Any face of a colored polytope is colored.
  Let P_λ be a colored polytope, f be a face of P_λ.
  There is an exact sequence

$$0 \longrightarrow W_f \xrightarrow{i} V_P \xrightarrow{q} V_P/W_f \longrightarrow 0,$$

  where $W_f = L(\{\lambda(F) \mid f \subset F \in \mathcal{F}\})$, $V_P = L(\{\lambda(F) \mid F \in \mathcal{F}\})$.
  Then there is an *induced colored map*

$$\lambda_f : \mathcal{F}_f \longrightarrow V_P/W_f$$

  such that $\lambda_f(s) = q \circ \lambda(F_s)$ for facet $s = f \cap F_s$ of f.

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  - For any colored polytopes P_1, P_2 containing f as a common face, the induced colored maps are the same and equal to λ_f.

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    - For any colored polytopes P_1, P_2 containing f as a common face, the induced colored maps are the same and equal to λ_f.
- A colored polytope is a colored polyhedral complex.

Colored homology of colored polyhedral complex

- *i*-th chain groups

$$C_i = \begin{cases} (\mathbb{Z}_2)^{|\mathcal{F}_0|}, & i = 0, \\ \oplus_{f \in \mathcal{F}_i} V_f, & i \geq 1, \end{cases}$$

where $\mathcal{F}_i$ is the *i*-face set of K_λ, any face $f \in \mathcal{F}_i$ is a colored polytope, V_f is the module generated by the image of λ_f.

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- Define boundary map $d_i : C_i \longrightarrow C_{i-1}$ via

$$d_i([f, v]) = \sum_{s \subset f} [s, q(v)]$$

where $s \in \mathcal{F}_f$ is a facet of f, and q is the quotient map in exact sequence

$$0 \longrightarrow W_s \xrightarrow{i} V_f \xrightarrow{q} V/W_s \longrightarrow 0.$$

- $d_i \circ d_{i+1} = 0$.

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- $H_i(K_\lambda; \mathbb{Z}_2) = \ker d_i / \text{im } d_{i+1}$ is called the *i-th colored homology group* of colored polyhedral complex K_λ, abbreviated as $H_i(K_\lambda)$.

Theorem

Let P_λ be a colored m-gon with edge set $\mathcal{E} = \{e_1, e_2, \cdots, e_m\}$ and colored map $\lambda : \mathcal{E} \longrightarrow (\mathbb{Z}_2)^k$ where $\lambda(e_i), \lambda(e_j)$ are linearly independent for $e_i \cap e_j \neq \varnothing$. Then

$$H_1(P_\lambda) = L(h_1, h_2, \cdots, h_m)/L(\{R_v \mid v \in (\mathbb{Z}_2)^k - \{0\}\})$$

where $h_i = e_i + e_i'$, $R_v = \sum_{\lambda(e_i) \neq v} h_i$ for $v \in (\mathbb{Z}_2)^k - \{0\}$.

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Corollary

Let P_λ be a colored m-gon with edge set $\mathcal{E} = \{e_1, e_2, \cdots, e_m\}$ and colored map $\lambda : \mathcal{E} \longrightarrow (\mathbb{Z}_2)^2$ where $\lambda(e_i), \lambda(e_j)$ are linearly independent for $e_i \cap e_j \neq \varnothing$. Then

$$H_1(P_\lambda) = L(h_1, h_2, \cdots, h_m)/L(R_1, R_2)$$

where $h_i = e_i + e'_i$, $R_1 = \sum_{\lambda(e_i)\neq(1,0)} h_i$, $R_2 = \sum_{\lambda(e_i)\neq(0,1)} h_i$.

Proposition

Let P_λ be an n-dimensional colored polytope satisfying that $V = (\mathbb{Z}_2)^n$ and that the colors $\{\lambda(F_i) \mid f \in F_i\}$ for each face $f = F_1 \cap F_2 \cap \cdots \cap F_k$ of P are linear independent. Then P_λ determines a small cover M. Moreover,

$$H_i(M; \mathbb{Z}_2) = H_i(P_\lambda).$$

The i-th betti number of P_λ equals to h_i, where h_i is the h-vector of the dual complex of simple polytope P.

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Remark

Let P_λ be an n-dimensional colored polytope satisfying that $V = (\mathbb{Z}_2)^k$ and that the colors $\{\lambda(F_i) \mid f \in F_i\}$ for each face $f = F_1 \cap F_2 \cap \cdots \cap F_k$ of P are linear independent. Then P_λ determines a closed n-manifold N. In general, $H_i(P_\lambda)$ is not isomorphic to $H_i(N; \mathbb{Z}_2)$.

Colored graph

- A graph $G = \{\mathcal{V}, \mathcal{E}\}$ is called a *colored graph* if there is a colored map on edge set $\mathcal{E}$,

$$\lambda : \mathcal{E} \longrightarrow V$$

where V is a $\mathbb{Z}_2$-module. Colored graph is denoted as G_λ.

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- If there is a map $\lambda : \mathcal{V} \longrightarrow V$ on vertex set of $G = \{\mathcal{V}, \mathcal{E}\}$, then the dual of G is a colored graph.

- The chain groups are defined as follows:

$$C_i = \begin{cases} (\mathbb{Z}_2)^{|\mathcal{V}|}, & i = 0, \\ \oplus_{e\in\mathcal{E}} V_e, & i = 1, \\ \oplus_{l\in\mathcal{L}} V_l, & i = 2, \\ 0, & otherwise, \end{cases}$$

where C_0 is generated by vertices of G_λ,
the generators of C_1 contains two types,

$$\begin{cases} \text{for } \lambda(e) \neq 0 \text{ the generator is edge } e \text{ and its copy } e',\ V_e = (\mathbb{Z}_2)^2, \\ \text{for } \lambda(e) = 0 \text{ the generator is edge } e,\ V_e = \mathbb{Z}_2, \end{cases}$$

$\mathcal{L}$ is the set of loops in G_λ and V_l is generated by $\{\lambda(e) \mid e \subset l\}$.

Homology groups of colored graph

- For any generator $e \in C_1$,

$$d_1(e) = v_1 + v_2,$$

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- For any $[l, h] \in C_2$,

$$d_2([l,h]) = \sum_{e_i \subset l, \lambda(e_i) = h} e_i + \sum_{e_i \subset l, \lambda(e_i) \notin \{0,h\}} e_i'.$$

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- For any $[I, h] \in C_2$,

$$d_2([I, h]) = \sum_{e_i \subset I, \lambda(e_i) = h} e_i + \sum_{e_i \subset I, \lambda(e_i) \notin \{0, h\}} e_i'.$$

- The difference with d_2 in colored homology of colored polytope is the definition of $d_2([I, 0])$.

$$\begin{cases} \text{Colored polytope: } d_2([I, 0]) = \sum_{e_i \subset I} e_i; \\ \text{Colored graph: } d_2([I, 0]) = \sum_{e_i \subset I, \lambda(e_i) = 0} e_i + \sum_{e_i \subset I, \lambda(e_i) \neq 0} e_i'. \end{cases}$$

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- $H_1(G_\lambda; \mathbb{Z}_2) = \ker d_1 / \mathrm{im}\ d_2$ is called the *(1-st) colored homology group* of colored graph G_λ, abbreviated as $H_1(G_\lambda)$.

Theorem

Let P_λ be a colored loop with edge set $\mathcal{E} = \{e_1, e_2, \cdots, e_m\}$ and colored map $\lambda : \mathcal{E} \longrightarrow (\mathbb{Z}_2)^k$ where $\lambda(e_i), \lambda(e_j)$ are linearly independent for $e_i \cap e_j \neq \varnothing$. Then

$$H_1(P_\lambda) = L(h_1, h_2, \cdots, h_m)/L(\{R_v \mid v \in (\mathbb{Z}_2)^k - \{0\}\})$$

where $h_i = e_i + e_i'$, $R_v = \sum_{\lambda(e_i)=v} h_i$ for $v \in (\mathbb{Z}_2)^k - \{0\}$.

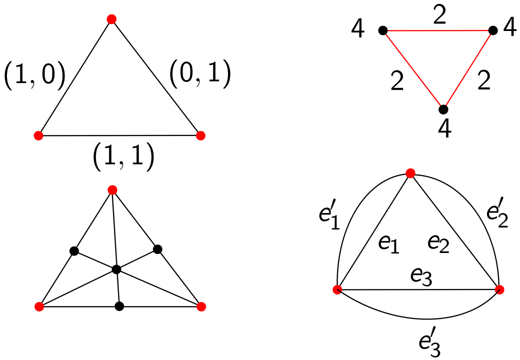
Corollary

Let P_λ be a colored loop with edge set $\mathcal{E} = \{e_1, e_2, \cdots, e_m\}$ and colored map $\lambda : \mathcal{E} \longrightarrow (\mathbb{Z}_2)^2$ where $\lambda(e_i), \lambda(e_j)$ are linearly independent for $e_i \cap e_j \neq \varnothing$. Then

$$H_1(P_\lambda) = L(h_1, h_2, \cdots, h_m)/L(R_1, R_2, R_3)$$

where $h_i = e_i + e_i'$, $R_1 = \sum_{\lambda(e_i)=(1,0)} h_i$, $R_2 = \sum_{\lambda(e_i)=(0,1)} h_i$, $R_3 = \sum_{\lambda(e_i)=(1,1)} h_i$.

Example



Let P_λ be a colored triangle.

- Weighted homology of dual complex $H_1^{wt}(K_P^w) \cong (\mathbb{Z}_2)^3$.
- st-homology $H_1^{st}(P_\lambda) \cong (\mathbb{Z}_2)^3$.
- Colored homology of colored triangle $H_1^{c1}(P_\lambda) \cong \mathbb{Z}_2$.
- Colored homology of colored loop $H_1^{c2}(P_\lambda) = 0$.

- Let G_λ^w be a weighted colored graph where graph $G = \{\mathcal{V}, \mathcal{E}\}$, colored map $\lambda : \mathcal{E} \longrightarrow V$ and weighted map $w : \mathcal{E} \longrightarrow \mathbb{R}$.

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- Fixed $d \geq 0$, let

$$\mathcal{E}(d) = \{e \mid w(e) \leq d, e \in \mathcal{E}\}$$

$$\lambda(d) = \lambda|_{\mathcal{E}(d)}.$$

Then $G_\lambda(d)$ with edge set $\mathcal{E}(d)$ is a colored subgraph of G_λ.

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  Then $G_\lambda(d)$ with edge set $\mathcal{E}(d)$ is a colored subgraph of G_λ.
- Making d change from 0 to ∞, the colored homology groups $\{H_1(G_\lambda(d)) \mid d \geq 0\}$ are called *persistent colored homology groups (PCH)* of weighted colored graph of G_λ^w.

Three adenine (A), EA, and ϵA

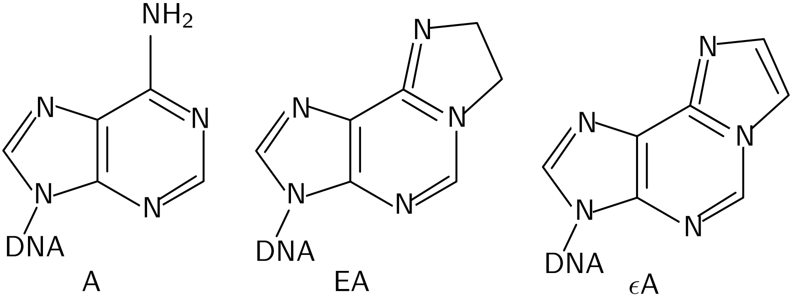

Figure: Structures of adenine (A), EA, and ϵA

Chemicalbond	Bondlength/$(10^{-12}$m)	Bondenergy/(KJ/mol)
C—C	154	332
C═C	134	611
C—H	109	414
C—N	148	305
C═N	135	615
N—H	101	389

- The color of each bond.

$$\lambda : \mathcal{E} \longrightarrow (\mathbb{Z}_2)^6,$$

$$\lambda(e) = \begin{cases} (1,0,0,0,0,0), & \text{if } e \text{ is bond C—C}, \\ (0,1,0,0,0,0), & \text{if } e \text{ is bond C=C}, \\ (0,0,1,0,0,0), & \text{if } e \text{ is bond C—H}, \\ (0,0,0,1,0,0), & \text{if } e \text{ is bond C—N}, \\ (0,0,0,0,1,0), & \text{if } e \text{ is bond N=N}, \\ (0,0,0,0,0,1), & \text{if } e \text{ is bond N—H}. \end{cases}$$

Colored map and weighted map

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- The weight of each bond $e \in \mathcal{E}$ can be take bond length or bond energy of e.

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- The weight of each bond $e \in \mathcal{E}$ can be take bond length or bond energy of e.
- All three adenines (A), EA, and ϵA can be modelled by weighted colored graph.

Calculation results—base on bond length

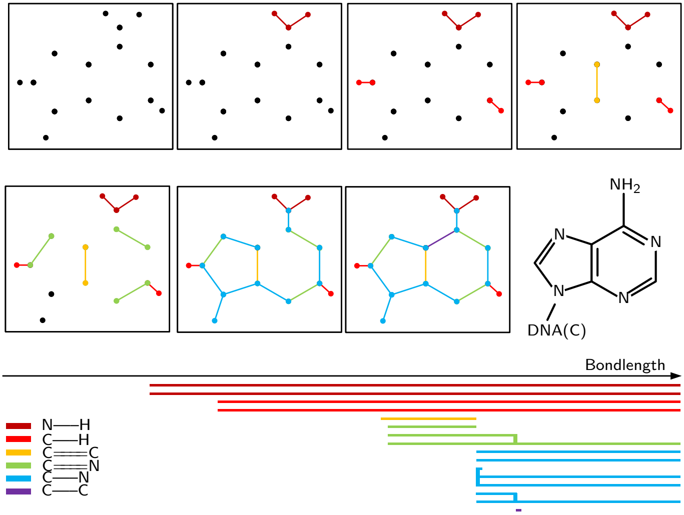

Figure: PCH of adenine (A)

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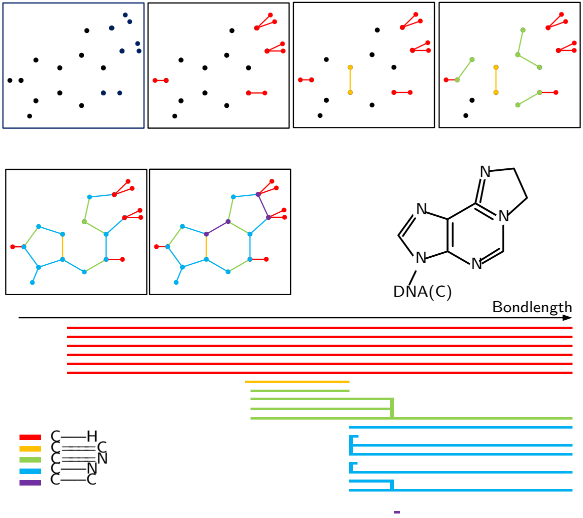


Figure: PCH of adenine EA

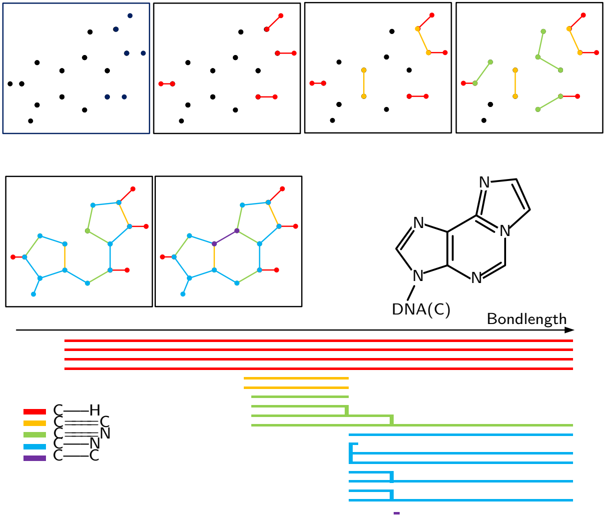


Figure: PCH of adenine ϵA

Calculation results—base on bond energy

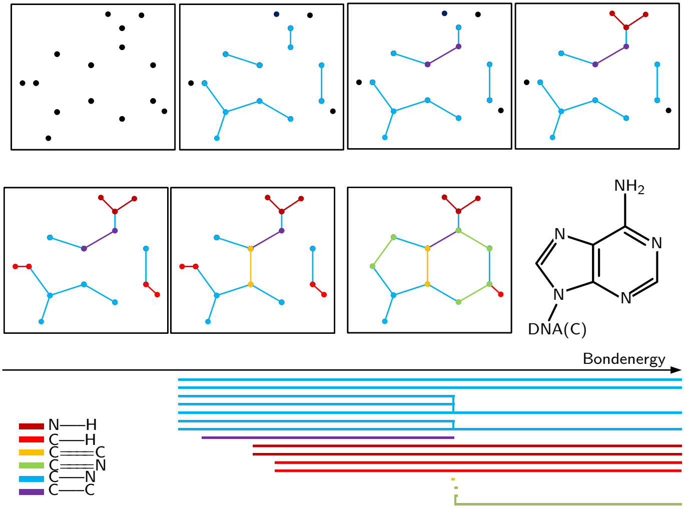


Figure: PCH of adenine (A)

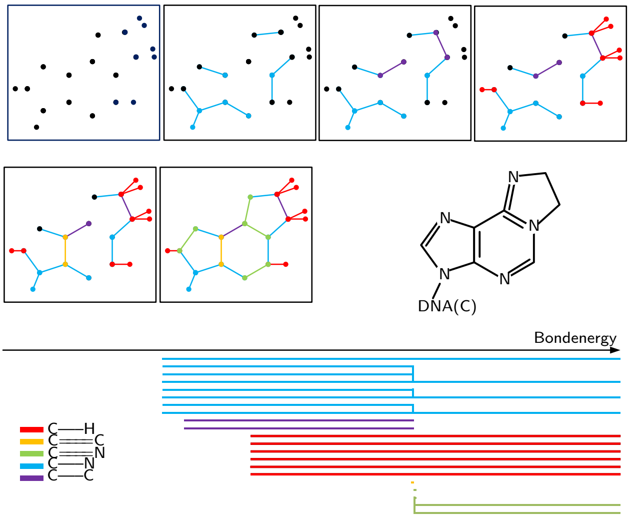


Figure: PCH of adenine EA

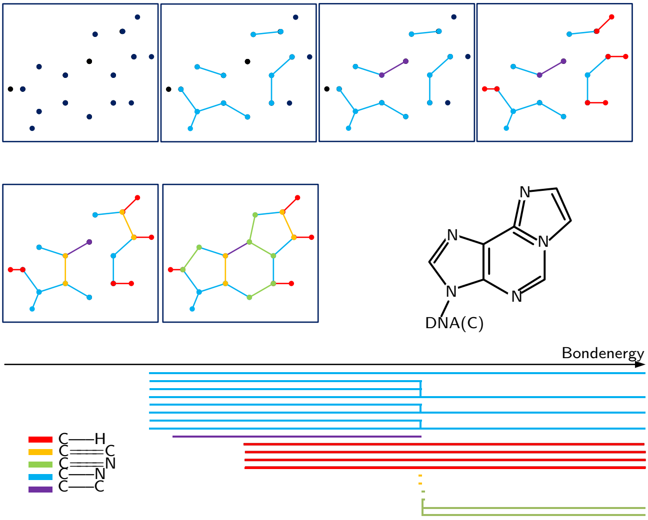
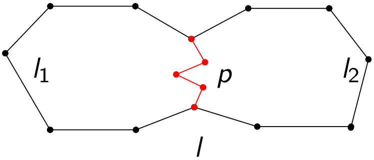
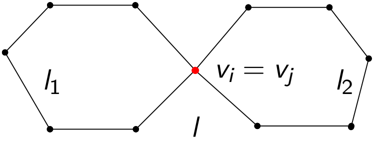


Figure: PCH of adenine ϵA

Computation of homology groups of colored graph

- Let l be a loop in graph G, $v_1, v_2, \cdots, v_{n+1}$ be the ordered vertices of l where $v_1 = v_{n+1}$. If there are two vertices v_i, v_j connected by a path p where p and l have no common edge, then l can be *split* into two loops l_1, l_2 which have common p. Specially, if there are two nonadjacent vertices v_i, v_j of l are the same vertex, then l can also be *split* into two loops l_1, l_2 which have common vertex $v_i = v_j$.





- The loop l with ordered vertices $\{v_1, v_2, v_1\}$ and one edge is called a *trivial loop*. Moreover, a loop which can be split into trivial loops is also called a *trivial loop*.

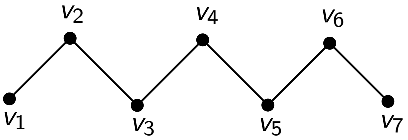


Figure: Trivial loop $v_1 v_2 \cdots v_6 v_7 v_6 \cdots v_2 v_1$

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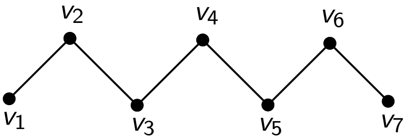
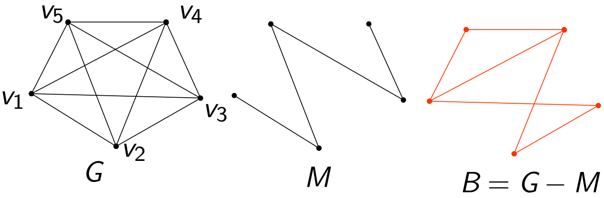
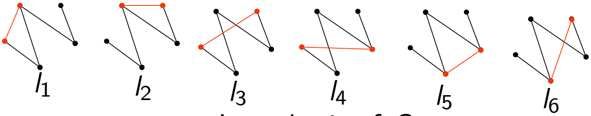


Figure: Trivial loop $v_1 v_2 \cdots v_6 v_7 v_6 \cdots v_2 v_1$

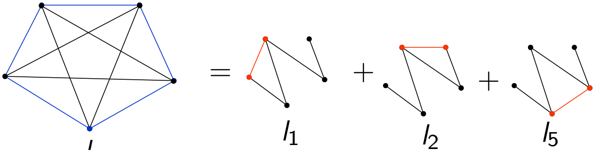
- Let M be a maximum spanning tree of graph G. Then adding an arbitrary edge $e \notin M$ on M will create a loop l_e. The loops $\mathcal{L}_b = \{l_e \mid e \notin M\}$ are called a *loop basis* of graph G, $l_e \in \mathcal{L}_b$ is called a *basis loop*.

- Any loop l of G can be split into basis loops in $\mathcal{L}_b$ and trivial loops.





Loop basis of G



- If loop l can be split into two loops l_1, l_2, then for $[l, h] \in C_2 = \oplus_{l \in \mathcal{L}} V_l$,

$$d_2([l, h]) = d_2([l_1, h]) + d_2([l_2, h]).$$

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Computation of homology groups of colored graph

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Proposition

Let $C_2' = \oplus_{l \in \mathcal{L}_b} V_l$, $d_2' = d_2 \mid_{C_2'}$. Then $C_2' \subset C_2$,

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- The loops set $\mathcal{L}$ of G_λ in the chain complex $X = \{C_i, d_i\}$ of colored graph G_λ can be replaced by loop basis $\mathcal{L}_b$.

Thanks for your attention!

Workshop on Toric Topology 2024 in Shanghai

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